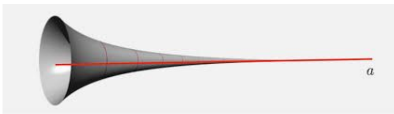
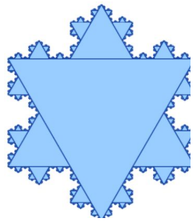
Can you draw a shape with infinite perimeter?



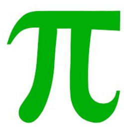


Math has shapes that can't be drawn because you would need to use infinitely many pencils. However they can fit on a single piece of paper!

How do you add infinitely many numbers?

By the time the hare catches up to the tortoise, the tortoise has gotten farther ahead. By the time the hare closes this gap, the tortoise has gotten farther ahead again. How can the Hare every catch up?

Will a monkey randomly typing forever write all of STAR WARS?

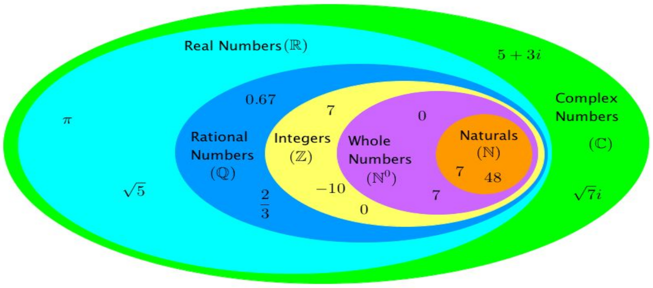


With enough random letters, every word, every sentence, and even any full book will be written! Some think the digits of pi are like this, can you find special numbers in pi?

Why does an infinite hotel have free rooms when it is full?

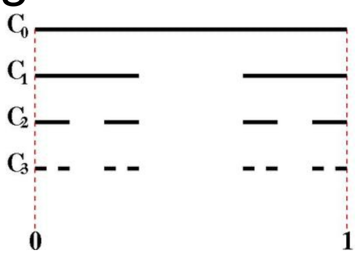
We can move people between rooms to create more space. Surprisingly, we can fit infinitely many buses each with infinitely many people!

How many different infinities are there?



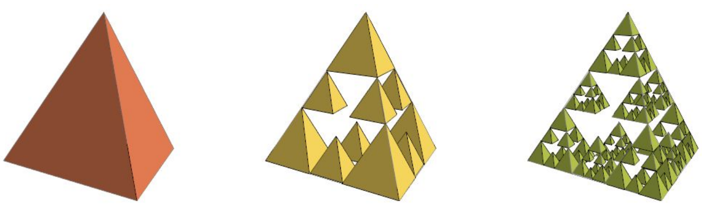
Numbers can be categorized into different sets that are infinite size. There are mainly two types: countable and uncountable. Which sets are countable and uncountable?

The Devil's Staircase: a never ending staircase of size 0



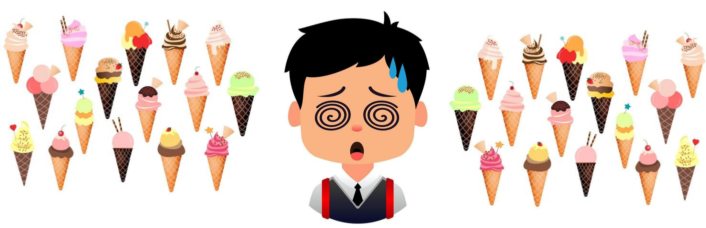
Start with an interval from 0 to 1. Remove the middle third. Then remove the middle third of the remaining two pieces. Then again and again. What is the size of this set as n goes to infinity?

Infinity stone - fractals in 3-dim



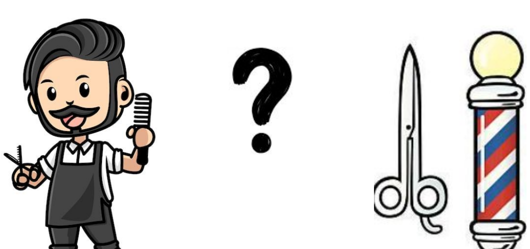
A fractal is a never ending pattern that allows us to capture some notion of infinity geometrically. They behave differently from the usual Euclidean geometric objects that we are used to.

Axiom of choice - Can you choose from infinite many choices?



Axiom of choice is a fundamental rule in mathematics that allows us to make choices from an infinite set.
It may sound absurd and obvious, but our math would break without this axiom!

The barber paradox ∞



Everyone in the village either shave themselves or are shaved by a barber, who also lives in the village.
The barber claims to shave only the villagers who do not shave themselves.
So who shaves the barber?

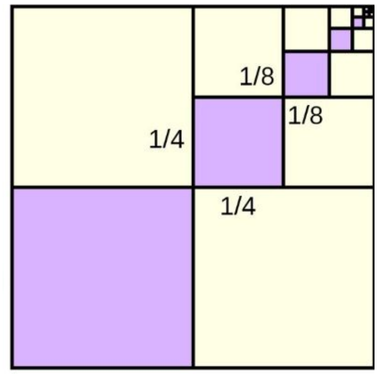
Fratals

Infinite Series

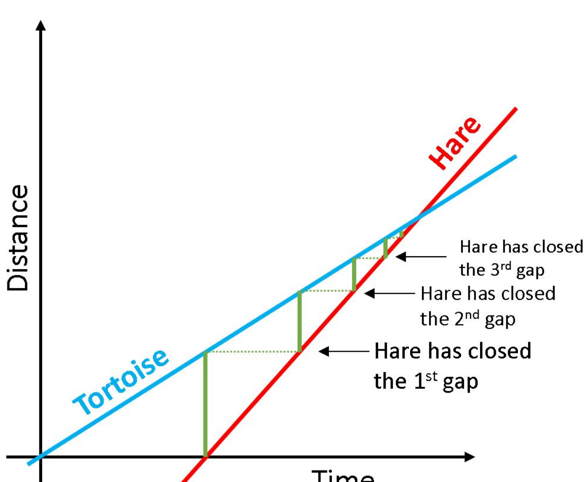
$$H = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \cdots$$

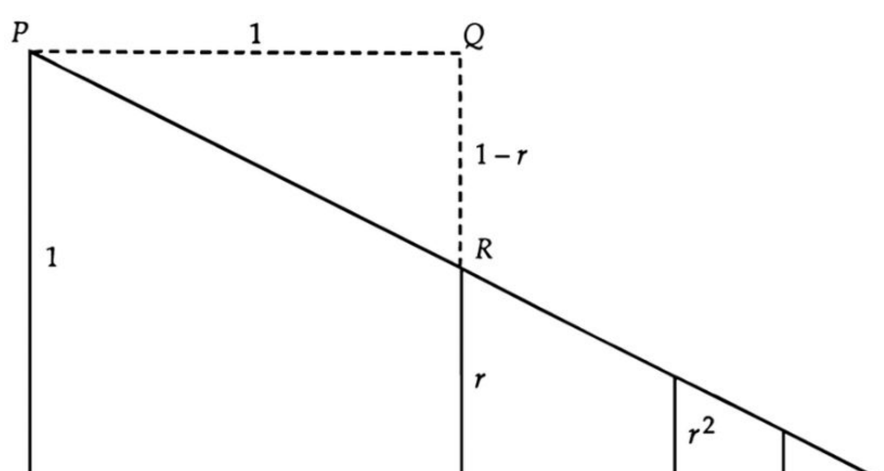
$$< \underbrace{\frac{1}{2} + \frac{1}{2}} + \underbrace{\frac{1}{4} + \frac{1}{4}} + \underbrace{\frac{1}{6} + \frac{1}{6}} + \underbrace{\frac{1}{8} + \frac{1}{8}} + \underbrace{\frac{1}{10} + \frac{1}{10}} + \cdots$$

$$H = 1 \quad + \quad \frac{1}{2} \quad + \quad \frac{1}{3} \quad + \quad \frac{1}{4} \quad + \quad \frac{1}{5} + \cdots$$



- We can write 1+1=2 or 1+1+1=3, but what happens if the sum continues forever? Strange things occur!
- Imagine a hare racing a tortoise with a head start. Every time the hare closes the gap, the tortoise has gotten a bit ahead, creating another gap. Therefore the hare can never pass the tortoise. This violation of common sense is one of **Zeno's paradoxes of motion**. The resolution is that this infinite sum of times converges.
- How much money would you pay to play the following game: you flip a fair coin until it comes up tails, and you winnings get doubled for each heads you land. Can do in person. A reasonable value for this is 1(½)+2(¼)+4(⅛)+... = ½+½+½+½+... which is infinite, so you should pay everything you have to play this game. But this is nonsensical, as it is almost guaranteed you won't win! This is known as the **St. Petersburg paradox**. One resolution is that utility
- The **Harmonic Series** is 1+½+⅓+¼+..., what number do you think this is? Even though the numbers shrink to zero, their sum blows up to infinity! This is because we can shrink each term by a little bit
- Avoid 1+2+3+4+... = -1/12 (https://prateekvjoshi.com/2014/01/24/1-2-3-4-5-112/)
- What value does 1-1+1-1+1-1+1-1+... have? Grouping terms starting at the first we get 0+0+0+0+..., so it should be zero. But grouping terms starting from the second gives 1+0+0+0+... = 1. This is the **Grandi Series**.
- Sometimes playing around with sums causes bigger issues where an infinite sum can become any value! This is the **Riemann Rearrangement Theorem**.
- Sometimes nothing weird happens, for example a+r+r^2+r^3+... = a/(1-r), geometric proof below (too complex?). This can be shown algebraically by solving rS = S-a since shifts the series is the same as removing the first term.
- These are **limits** of partial sums. Sometimes the limit does not exist (Grandi's).

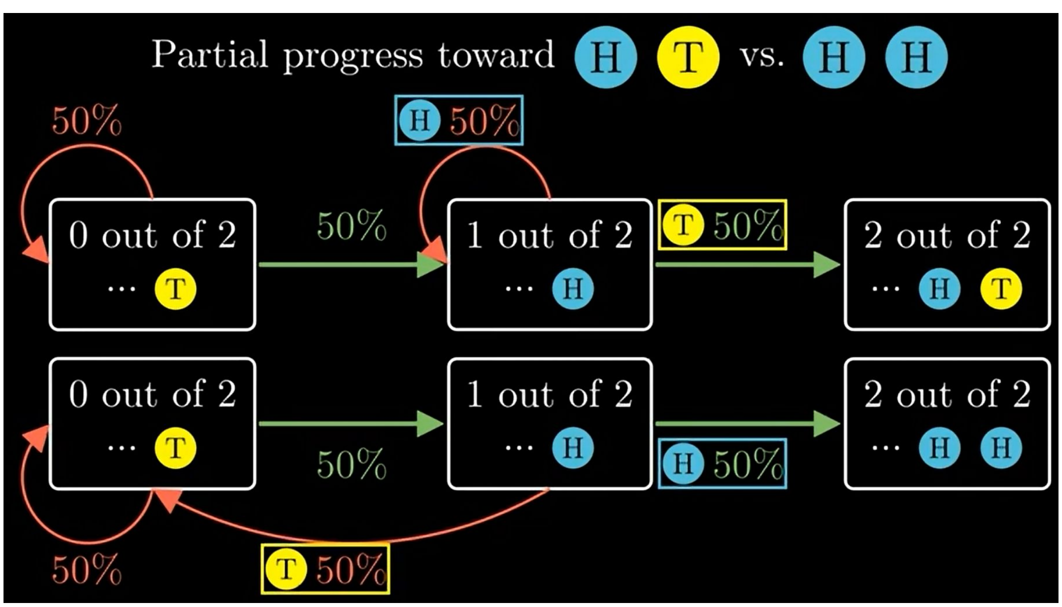




$$\mathbb{E}[N_{a_1 a_2 \ldots a_k}] = \left(\frac{1}{p}\right)^k + \sum_{\substack{1 \leq i \leq k-1 \text{ such that} \\ a_1 \ldots a_i = a_{n-i+1} \ldots a_n}} \left(\frac{1}{p}\right)^i$$

Abracadabra Theorem

- Given infinite time, can anyone write a best selling novel? What if they just randomly types keys?
- Surprisingly, the answer is yes! Proof intuition - every block of N letters, there is some non-zero chance the goal N letters will be spelled. As we repeat these blocks over and over again, the probability that NONE of them is the goal goes to zero. “No matter how rare something is, if you keep trying, you’ll get it” <- assuming independence…
- Some people think **irrational numbers** like pi act like random monkeys on a typewriter. Can you find meaningful sequences at https://www.subidiom.com/pi/ ? While we know the digits go on forever and never repeat, this doesn’t guarantee all sequences will occur. For example, 1011011101111011111011111… never repeats but also misses the digits 2-9. An additional property called “normality” guarantees each digit appears equally often (this is also not necessary for representing all sequences, as they need not occur equally often)
- We know all sequences will appear, but how long will it take? Let’s pretend we have a really simple typewriter, only having heads and tails.The probability a block matches a goal is the same for all goals, but not if one considers all blocks within a longer sequence, since there are overlaps in the blocks so they are not independent,
- So easy sequences are have minimal overlap like XXX, and hard sequences have a lot of overlaps. Example table for 3 or 4 flips?
- If you want to trick your friends, you can argue that any sequence is equally likely since the die is fair, but choose a sequence that is likely to show up sooner.



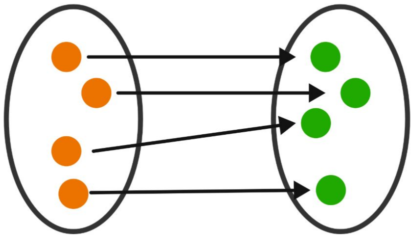
Hilbert’s Hotel

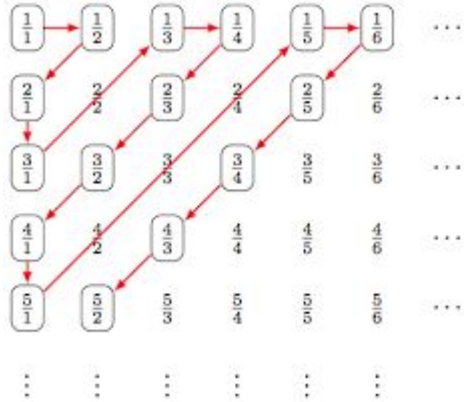
How would you fit…	**Hint / Solution**	**How does this extend to**
One more guest?	Once you move the first one to room two, where should the one in room two go / what happens after this?	10 or 100 guests?
A bus with infinitely many new guests?	How do people make room when there is space, they spread out. Can the hotel occupants spread out too?	Two or 10 buses?
Infinitely many buses, each with infinitely many new guests?	Each bus needs to be more and more spread out, how can we achieve this with arithmetic operations? We used multiples before, will powers work?	

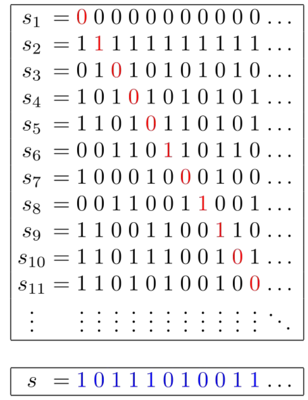
- Write on poster (with representative drawings): Imagine we had an infinitely large hotel that is full, how would we make space for … (write questions below as 1, 2, 3).
- To make the last one easier, imagine the hotel is empty at first and it’s original occupants are in the first bus. How can we fill this hotel with each bus so that none of the buses send people to the same room?
- The uniqueness of **prime factorization** shows overlaps with powers won’t occur!
-
- If solved all: How full is our hotel? Quite sparse actually, as any number not in the form p^k will not be hit. It looks like we have a lot of space left after all!

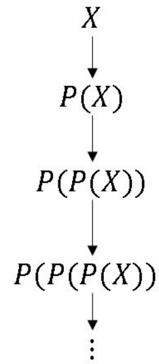
To make: A representation of boxes with room numbers, e.g. into a 5 by 10 grid with … to show it continues beyond this. Participants move around chip with colors representing the bus of origin to get a physical intuition for how people move. Several should be prepared so many can play at once. Questions on poster, ask participants to share their solution or ask for hints if they feel stuck. They can hear the answer and be asked to think how it would generalize.

Countability









- **Frame as a puzzle**: Do N, Z, Q, or R have the same size? Show even = N first. More relatable, subsets of toppings for an infinite sundae, names (finite) as letter sequences.
- How can we prove two infinities are different? One way to define infinity, or any number, is as the sizes of a **"set"**, a collection of elements like {apple, pear, banana}. Numbers are the same if we can create a 1:1 map between some sets for each, as shown in the picture above.
- We can do this with infinity too. For example, the set of natural numbers N={1, 2, 3, 4, 5, …} has the same size as the set of integers Z={..., -3, -2, -1, 0, 1, 2, 3, …} because we can write them all out in order, e.g. 0, 1, -1, 2, -2, 3, -3, … When we can do this, we say the set is **"countable"** because we can count it out. Rational numbers are countable, can you show this? The above picture is one way. Similarly, the set of names beginning with A has the same size as all names.
- Real numbers are another infinite set but are not countable. In 1891, Georg Cantor made a Diagonal argument to prove this. Any real number is given by an infinite sequence of digits. Suppose we could list all real numbers, as shown above. Then we could construct a real number not in our list by changing the diagonal entries, the kth digit of entry k in our list. This guarantees our new number is not any entry in the list. Since we always miss a real number, any list we attempt will be incomplete. This is known as a proof by contradiction, where one shows an assumption creates a contradiction, so the assumption must be false. How many numbers did we miss? There are 9 choices for different digits, so we missed uncountably many!
- Given any set, it's **"Power Set"** is the set of all subsets, for example X = {1, 2} has power set P(X) = {empty, {1}, {2}, {1, 2}}. Each element can independently be included or not, so multiplying all these two choices shows |P(X)| = 2^|X|. The binary representation of real numbers is a 1:1 map between the real numbers and subsets of the natural numbers, where the subsets corresponds to the positions of the 1's in the binary representation.
- We can generalize Cantor's argument to show |X| < |P(X)|. To show less than or equal to, we send x to {x}, to show inequality, assume that there was such a map f. Using f, construct the subset R of X given by the x that are not in the sets f(x) they are mapped to. Since f is 1:1, R=f(r) for some r in X. Is r in R? Either way leads to a contradiction, so no 1:1 map can exist. Starting with any countable set and repeatedly taking powers of it gives us a (countable) number of infinities! This is shown at the far right above. The **continuum hypothesis** asked if there were any infinities between |N| and |P(N)|, it turns out to be independent of commonly chosen axioms, or base assumptions, for mathematics.
- This is known as the **"Barber's Paradox"**, invented by Betrand Russell in 1901 to show potential issues with set theory, which was proposed as a basis for mathematic. If X is the set of people and f gives the subset of people shaved by a given person, then R is the set of people who don't have themselves, and r is a barber who doesn't shave themselves. Whether r is in R is "Does a barber who shaves precisely those who don't shave themselves shave themself?"
- This has applications to the **"halting problem"** / computability theory.
- Are there any other infinities? This is known as the "continuum hypothesis" and It turns out to be independent from the common "axioms" taken for math.
- The **"ordinal numbers"** define |N| as omega and show how to treat it like a number, giving meaning to expressions like omega+1 and omega^2+3*omega. Also some **axiom of choice** weirdness.

Connections

to->	Fractals	Infinite sums	Abracadbra	Hilbert hotel	countability
Fractals	X				
sums	Perimeter = infinity, area = finite	X			
monkeys	nesting?		X		
hotel	Maybe nesting			X	
countability	Cantor set,		All sequences are monkey outputs	Works because all countable, there are others	X